MATHEMATICS

ON IDEAL THEORY IN GENERAL RINGS

BY

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Introduction

A systematical study of prime ideals in general rings was made by N. H. McCOY [5]. Before this some properties of prime ideals were discussed by KRULL [2] and FITTING [1]. In a *commutative* ring R an ideal $\mathfrak{p}$ is a prime ideal if and only if $ab \equiv 0(\mathfrak{p})$ implies $a \equiv 0(\mathfrak{p})$ or $b \equiv 0(\mathfrak{p})$. For a noncommutative ring this definition is too strong to be of much interest. In this case it is customary to call an ideal $\mathfrak{p}$ a prime ideal if and only if $\mathfrak{a}\mathfrak{b} \equiv 0(\mathfrak{p})$ implies that $\mathfrak{a} \equiv 0(\mathfrak{p})$ or $\mathfrak{b} \equiv 0(\mathfrak{p})$, where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R. For general rings we use this definition, an ideal which is prime in the first sense is called completely prime. An ideal which is completely prime is prime, but the converse is not generally true. However, these concepts coincide in the case of commutative rings. An equivalent definition is: an ideal $\mathfrak{p}$ in an arbitrary ring R is a prime ideal if and only if $axb \equiv 0(\mathfrak{p})$ for all $x \in R$ implies that $a \equiv 0(\mathfrak{p})$ or $b \equiv 0(\mathfrak{p})$. In view of this definition McCoy introduces the concept of m-system M as a system of elements of R with the property that $c, d \in M$ imply the existence of an element $x \in R$ such that $cxd \in M$. Then an ideal $\mathfrak{p}$ is a prime ideal if and only if the complement of $\mathfrak{p}$ is an m-system.

KRULL [3] has developed a theory of ideals in commutative rings. Using the above definition of completely prime ideal $\mathfrak{p}$ he remarks that the complement of $\mathfrak{p}$ is a multiplicatively closed system. It is the purpose of this paper to extend to general, that is, not necessarily commutative rings several results of Krull for commutative rings. Instead of his completely prime ideals we use prime ideals according to the definition of McCoy, such that his multiplicatively closed system is replaced by our m-system. We start with the definition of an element related to an ideal. In a previous paper [4] we used another definition of this concept, which is less useful. In the literature, one finds "b is prime (not prime) to $\mathfrak{a}$" in place of our "b is unrelated (related) to $\mathfrak{a}$". The use of "related to" is in accordance with McCOY [6]. This leads to the definition of prime ideal (definition 4), which is, in a sense, an element-definition. The remaining part of § 1 is devoted to the introduction of the maximal ideals belonging to an ideal $\mathfrak{a}$ as the ideals which are maximal with respect to the property of being related to $\mathfrak{a}$. In § 2 the principal components of an ideal $\mathfrak{a}$ are defined, which have similar properties as in the commutative case. Generalizing, we obtain the isolated component ideal of an ideal $\mathfrak{a}$ with the aid of the concept of m-system (§ 3). Our main result is: an ideal $\mathfrak{a}$ is the intersection of all its principal r-components $\mathfrak{a}_r(\mathfrak{p})$ or of its principal l-components $\mathfrak{a}_l(\mathfrak{p}')$, (theorem 3).

1. *Maximal prime ideals belonging to an ideal.*

Let R be an associative, though not necessarily commutative ring. In R we consider only two-sided ideals. Then we define:

Definition 1. An element a of R is *left related to* the ideal $\mathfrak{a}$ (l-related to $\mathfrak{a}$), in case there exists an element b *not in* $\mathfrak{a}$ such that $axb \equiv 0(\mathfrak{a})$ for all $x \in R$.

An element a of R is *right related to* $\mathfrak{a}$ if there exists an element b' *not in* $\mathfrak{a}$ such that $b'ya \equiv 0(\mathfrak{a})$ for all $y \in R$.

Definition 2. The element a is *left unrelated to* $\mathfrak{a}$ (l-unrelated to $\mathfrak{a}$) if $axb \equiv 0(\mathfrak{a})$ for an element b of R and all x of R implies $b \equiv 0(\mathfrak{a})$.

The element a is *right unrelated to* $\mathfrak{a}$ (r-unrelated to $\mathfrak{a}$) if $b'ya \equiv 0(\mathfrak{a})$ for an element b' of R and all y of R implies $b' \equiv 0(\mathfrak{a})$.

If c and d are l-unrelated to $\mathfrak{a}$, then there exists an element x' of R such that $cx'd$ is l-unrelated to $\mathfrak{a}$. Suppose cxd is l-related to $\mathfrak{a}$ for *all* $x \in R$. Then there exists an element t *not in* $\mathfrak{a}$ such that $cxdyt \equiv 0(\mathfrak{a})$ for all $y \in R$. But c is l-unrelated to $\mathfrak{a}$, so $dyt \equiv 0(\mathfrak{a})$ for all $y \in R$. It follows that, since d is l-unrelated to $\mathfrak{a}$, $t \equiv 0(\mathfrak{a})$. This is a contradiction, since t is not in $\mathfrak{a}$, therefore there exists at least one element $x' \in R$ such that $cx'd$ is l-unrelated to $\mathfrak{a}$.

Now we define (cf. McCoy [5]):

Definition 3. A set M of elements of R is an *m-system* if and only if $c \in M, d \in M$ imply that there exists an element x of R such that $cxd \in M$. The void set is to be considered as an m-system.

From the above it follows that the elements, which are l-unrelated to the ideal $\mathfrak{a}$, form an m-system M. Likewise the elements, which are r-unrelated to the ideal $\mathfrak{a}$, form an m-system M'. It may be pointed out that $M \neq M'$ in general. The importance of the concept of m-system lies in the fact that an ideal $\mathfrak{p}$ is a prime ideal if and only if its complement $S_{\mathfrak{p}}$ in R is an m-system [5]. We use the definition:

Definition 4. An ideal $\mathfrak{p}$ is called a *prime ideal* if $axb \equiv 0(\mathfrak{p})$ for all $x \in R$ implies $a \equiv 0(\mathfrak{p})$ or $b \equiv 0(\mathfrak{p})$; i.e. if $c \not\equiv 0(\mathfrak{p})$, then c is l-unrelated and r-unrelated to $\mathfrak{p}$.

According to this definition, the elements of $S_{\mathfrak{p}}$ form an m-system M. For if c and d are not in $\mathfrak{p}$, there always exists an element $x' \in R$ such that $cx'd \not\equiv 0(\mathfrak{p})$. Otherwise c or d would belong to $\mathfrak{p}$, supposing $\mathfrak{p}$ is a prime ideal. If $\mathfrak{p}$ is a prime ideal, we can prove that c is l-unrelated to $\mathfrak{p}$

implies c is r-unrelated to $\mathfrak{p}$ and conversely. In the trivial case in which $\mathfrak{p}=R$ we assume: no element of R is l-unrelated and r-unrelated to $\mathfrak{p}$. If $\mathfrak{p}\neq R$, suppose that c is l-unrelated to $\mathfrak{p}$ and r-related to $\mathfrak{p}$. Then there exists an element d'' not in $\mathfrak{p}$ such that $d''yc\equiv 0(\mathfrak{p})$ for all $y\in R$. It follows that $c\equiv 0(\mathfrak{p})$, as $\mathfrak{p}$ is a prime ideal, but then $cxd\equiv 0(\mathfrak{p})$ for all $x\in R$ and any element d of R. If $d\not\equiv 0(\mathfrak{p})$, c is l-related to $\mathfrak{p}$, which is a contradiction.

So we see, that c is l-unrelated to $\mathfrak{p}$ implies c is r-unrelated to $\mathfrak{p}$. Clearly, the converse is also true. The elements, which are l-unrelated to $\mathfrak{p}$ form an m-system M'; the elements, which are r-unrelated to $\mathfrak{p}$ form an m-system M''. We have proved that $M'=M''$. Now it is easy to see that $M=M'=M''$ also holds, if M is the m-system of elements of $S_{\mathfrak{p}}$. From definition 4 it follows, that $c\in M$ implies $c\in M'=M''$. But if $c\in M'$ or if c is l-unrelated to $\mathfrak{p}$, then c cannot belong to $\mathfrak{p}$, as all elements of $\mathfrak{p}$ are l-related to $\mathfrak{p}$, according to definition 1. Therefore $c\in S_{\mathfrak{p}}=M$. The agreement to consider the void set as an m-system is to take care of the special case in which $\mathfrak{p}=R$. We agree now, that in this case no element of R is l-unrelated or r-unrelated to $\mathfrak{p}$. Then $M'=M''$ is the void set. Henceforth, we assume $\mathfrak{p}\neq R$.

If two sets of elements of R have no elements in common, we say that either of these sets *does not meet* the other.

Definition 5. An *ideal* $\mathfrak{b}$ is *l-related to* $\mathfrak{a}$ if every element of $\mathfrak{b}$ is l-related to $\mathfrak{a}$.

An *ideal* $\mathfrak{c}$ is *r-related to* $\mathfrak{a}$ if every element of $\mathfrak{c}$ is r-related to $\mathfrak{a}$.

The ideal $\mathfrak{b}'$ is *l-unrelated to* $\mathfrak{a}$, if it is not l-related to $\mathfrak{a}$; the ideal $\mathfrak{c}'$ is *r-unrelated to* $\mathfrak{a}$, if it is not r-related to $\mathfrak{a}$.

We can prove the lemma [5], cf. also [4]:

Lemma. Let M be an m-system in R, and $\mathfrak{a}$ an ideal which does not meet M. Then $\mathfrak{a}$ is contained in an ideal $\mathfrak{p}$ which is maximal in the class of ideals which do not meet M. The ideal $\mathfrak{p}$ is necessarily a prime ideal.

Proof. The *existence* of $\mathfrak{p}$ with the required maximal property is assured by Zorn's lemma, applicated to the set of all ideals which contain $\mathfrak{a}$ but do not meet M. We show, that $\mathfrak{p}$ is a prime ideal. We assume, that $a\not\equiv 0(\mathfrak{p})$ and $b\not\equiv 0(\mathfrak{p})$ and show that there exists an element x' of R such that $ax'b\equiv 0(\mathfrak{p})$. The maximal property of $\mathfrak{p}$ implies that $(\mathfrak{p}, a)$ contains an element m_1 of M, and likewise $(\mathfrak{p}, b)$ contains an element m_2 of M.

Since M is an m-system, there is an element x of R such that $m_1xm_2\in M$, and hence $m_1xm_2\not\equiv 0(\mathfrak{p})$, since $\mathfrak{p}$ does not meet M. As $m_1\in(\mathfrak{p}, a)$ and $m_2\in(\mathfrak{p}, b)$ we have $m_1\equiv a_1(\mathfrak{p})$ and $m_2\equiv b_1(\mathfrak{p})$ with $a_1\in(a), b_1\in(b)$, and hence $a_1xb_1\equiv m_1xm_2(\mathfrak{p})$ and, therefore $a_1xb_1\not\equiv 0(\mathfrak{p})$. Now if $ayb\equiv 0(\mathfrak{p})$ for *all* $y\in R$, then also $a_1xb_1\equiv 0(\mathfrak{p})$, which does not hold; so there exists at least one element, say x', of R such that $ax'b\equiv 0(\mathfrak{p})$. This shows that $\mathfrak{p}$ is a prime ideal.

Since the m-system M, consisting of all elements which are l-unrelated to $\mathfrak{a}$, does not meet the ideal $\mathfrak{a}$ (every element of $\mathfrak{a}$ is l-related to $\mathfrak{a}$), the lemma asserts the existence of at least one ideal which is maximal in the class of all ideals which do not meet M, i.e. the class of ideals, l-related to $\mathfrak{a}$. Each such maximal ideal is a prime ideal.

Definition 6. An ideal which is maximal in the set of all ideals l-related to $\mathfrak{a}$ is called a *maximal l-prime ideal belonging to* $\mathfrak{a}$. An ideal which is maximal in the set of all ideals r-related to $\mathfrak{a}$ is called a *maximal r-prime ideal belonging to* $\mathfrak{a}$. Expressed otherwise, an ideal $\mathfrak{p}$ is a maximal l-prime resp. r-prime ideal belonging to $\mathfrak{a}$ if and only if $\mathfrak{p}$ is l-related resp. r-related to $\mathfrak{a}$, but any ideal $\mathfrak{n}$ such that $\mathfrak{p} \subset \mathfrak{n}$ is l-unrelated resp. r-unrelated to $\mathfrak{a}$.

If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two maximal l-prime ideals belonging to $\mathfrak{a}$, then $\mathfrak{p}_1 \subset \mathfrak{p}_2$ is impossible. Suppose $\mathfrak{p}_1 \subset \mathfrak{p}_2$. The ideal $\mathfrak{p}_1$ is l-related to $\mathfrak{a}$, but the ideal $\mathfrak{p}_2$ with $\mathfrak{p}_1 \subset \mathfrak{p}_2$ is l-unrelated to $\mathfrak{a}$. This is a contradiction, since $\mathfrak{p}_2$ is l-related to $\mathfrak{a}$ according to the definition 6. Clearly $\mathfrak{p}_1 \subset \mathfrak{p}_2$ or $\mathfrak{p}_2 \subset \mathfrak{p}_1$ is also impossible, if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two maximal r-prime ideals belonging to $\mathfrak{a}$.

We now show: $\mathfrak{a}$ *is contained in every maximal l-prime ideal* $\mathfrak{p}$ *belonging to* $\mathfrak{a}$. Consider the sum $(\mathfrak{a}, \mathfrak{p})$ of the ideals $\mathfrak{a}$ and $\mathfrak{p}$, and let $a+p$ be any element of this sum, where $a \in \mathfrak{a}$, $p \in \mathfrak{p}$. Since $\mathfrak{p}$ is l-related to $\mathfrak{a}$, we have $pxr \equiv 0(\mathfrak{a})$ for all $x \in R$ and some r, not in $\mathfrak{a}$. Hence $(a+p)xr \equiv 0(\mathfrak{a})$, that is, $a+p$ is l-related to $\mathfrak{a}$. As $a+p$ is an arbitrary element of $(\mathfrak{a}, \mathfrak{p})$, it follows that $(\mathfrak{a}, \mathfrak{p})$ is l-related to $\mathfrak{a}$, and since $\mathfrak{p} \subseteq (\mathfrak{a}, \mathfrak{p})$, the maximal property of $\mathfrak{p}$ shows $\mathfrak{p} = (\mathfrak{a}, \mathfrak{p})$. This, however, implies that $\mathfrak{a} \subseteq \mathfrak{p}$, which completes the proof. In the same way it may be shown that $\mathfrak{a}$ is contained in every maximal r-prime ideal $\mathfrak{p}^*$ belonging to $\mathfrak{a}$.

Theorem 1. Every element or ideal, l-related to $\mathfrak{a}$, is contained in a maximal l-prime ideal belonging to $\mathfrak{a}$.

Proof. If b is l-related to $\mathfrak{a}$, then (b), the principal ideal generated by b, is l-related to $\mathfrak{a}$. For if $bxr \equiv 0(\mathfrak{a})$ for all $x \in R$ and some r not in $\mathfrak{a}$, then $b_1xr \equiv 0(\mathfrak{a})$ for all $x \in R$ and any element b_1 of (b). Therefore (b) is l-related to $\mathfrak{a}$. Hence, in the proof of the theorem, the only case which need be considered is that of an *ideal* which is l-related to $\mathfrak{a}$.

If $\mathfrak{b}$ is an ideal l-related to $\mathfrak{a}$, the m-system M, which consists of the elements of R l-unrelated to $\mathfrak{a}$, contains no element of $\mathfrak{b}$. The lemma then shows the existence of a maximal l-prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$ such that $\mathfrak{b} \subseteq \mathfrak{p}$. It may be remarked that every element or ideal, r-related to $\mathfrak{a}$, is contained in a maximal r-prime ideal belonging to $\mathfrak{a}$.

Theorem 1 makes it clear, that all elements l-related resp. r-related to $\mathfrak{a}$ are spread over the maximal l-prime resp. r-prime ideals belonging to $\mathfrak{a}$.

2. *The principal component of an ideal.*

Now let $\mathfrak{p}$ be any maximal r-prime ideal belonging to $\mathfrak{a}$ and $S_\mathfrak{p} = R - \mathfrak{p}$, so that $S_\mathfrak{p}$ consists of all the elements of R, which are not in $\mathfrak{p}$.

Definition 7. The ideal consisting of all and only those elements c of R, such that there exists an element s in $S_{\mathfrak{p}}$ with $cxs \equiv 0(\mathfrak{a})$ for all $x \in R$, is called the *principal r-component* $(\mathfrak{a}_{S_{\mathfrak{p}}})_r = \mathfrak{a}_r(\mathfrak{p})$ *of* $\mathfrak{a}$.

In the special case that $\mathfrak{p} = R$, we have $S_{\mathfrak{p}}$ is the void set and we define $(\mathfrak{a}_{S_{\mathfrak{p}}})_r = \mathfrak{a}$.

Likewise we may define: d' belongs to *the principal l-component* $(\mathfrak{a}_{S_{\mathfrak{p}'}})_l = \mathfrak{a}_l(\mathfrak{p}')$ *of* $\mathfrak{a}$, where $\mathfrak{p}'$ is any maximal l-prime ideal belonging to $\mathfrak{a}$, if and only if there exists an element s' of $S_{\mathfrak{p}'}$ with $s'yd' \equiv 0(\mathfrak{a})$ for all $y \in R$. We agree: $(\mathfrak{a}_{S_{\mathfrak{p}'}})_l = \mathfrak{a}$, in case $\mathfrak{p}' = R$.

The ideal $\mathfrak{a}$ is contained in $\mathfrak{a}_r(\mathfrak{p})$ resp. $\mathfrak{a}_l(\mathfrak{p}')$; for $axs \equiv 0(\mathfrak{a})$ resp. $s'ya \equiv 0(\mathfrak{a})$ for all $x, y \in R$, $s \in S_{\mathfrak{p}}$ resp. $s' \in S_{\mathfrak{p}'}$ and any $a \in \mathfrak{a}$. But $\mathfrak{a}_r(\mathfrak{p})$ resp. $\mathfrak{a}_l(\mathfrak{p}') \neq R$. If $\mathfrak{p} = R$, we have $\mathfrak{a}_r(\mathfrak{p}) = \mathfrak{a} \neq R$; likewise if $\mathfrak{p}' = R$, we have $\mathfrak{a}_l(\mathfrak{p}') = \mathfrak{a} \neq R$. If $\mathfrak{p} \neq R$, suppose that $\mathfrak{a}_r(\mathfrak{p}) = R$. Let c be an arbitrary element of $S_{\mathfrak{p}}$, then $c \in \mathfrak{a}_r(\mathfrak{p}) = R$. This means, there exists an element s in $S_{\mathfrak{p}}$ such that $cxs \equiv 0(\mathfrak{a})$ for all $x \in R$. Therefore $cxs \equiv 0(\mathfrak{p})$ for all $x \in R$ with $c, s \in S_{\mathfrak{p}}$, which is a contradiction since $\mathfrak{p}$ is a prime ideal. Therefore $\mathfrak{a}_r(\mathfrak{p}) \neq R$ and similarly we may show $\mathfrak{a}_l(\mathfrak{p}') \neq R$.

The elements of $S_{\mathfrak{p}}$ *are r-unrelated to* $\mathfrak{a}_r(\mathfrak{p})$. Indeed, let s be an element of $S_{\mathfrak{p}}$ and $rxs \equiv 0(\mathfrak{a}_r(\mathfrak{p}))$ for all $x \in R$. Then there exists an element t in $S_{\mathfrak{p}}$ such that $rxsyt \equiv 0(\mathfrak{a})$ for all $x, y \in R$. As $S_{\mathfrak{p}}$ is an m-system $s \in S_{\mathfrak{p}}, t \in S_{\mathfrak{p}}$ imply the existence of an element y_1 of R such that $sy_1t \in S_{\mathfrak{p}}$ (definition 3). Then $rxsy_1t \equiv 0(\mathfrak{a})$ for all $x \in R$ and $r \in \mathfrak{a}_r(\mathfrak{p})$, according to definition 7. This means, s is r-unrelated to $\mathfrak{a}_r(\mathfrak{p})$. In a similar way we can prove that the elements of $S_{\mathfrak{p}'}$ are l-unrelated to $\mathfrak{a}_l(\mathfrak{p}')$.

Suppose now, that there exist at least two different maximal r-prime ideals belonging to $\mathfrak{a}$: $\mathfrak{p}_1$ and $\mathfrak{p}_2$. As we have seen, $\mathfrak{p}_1 \not\subset \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subset \mathfrak{p}_1$, so $\mathfrak{p}_2$ contains at least one element, say p_2, not in $\mathfrak{p}_1$ or $p_2 \in S_{\mathfrak{p}_1}$. Therefore p_2 is r-unrelated to $\mathfrak{a}_r(\mathfrak{p}_1)$. If $\mathfrak{a}_r(\mathfrak{p}_1) = \mathfrak{a}$, p_2 is r-unrelated to $\mathfrak{a}$. It follows that $\mathfrak{p}_2$ is r-unrelated to $\mathfrak{a}$, which is a contradiction, since $\mathfrak{p}_2$ is a maximal r-prime ideal belonging to $\mathfrak{a}$. Therefore, if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two different maximal r-prime ideals belonging to $\mathfrak{a}$, $\mathfrak{a} \subset \mathfrak{a}_r(\mathfrak{p}_1)$ and $\mathfrak{a} \subset \mathfrak{a}_r(\mathfrak{p}_2)$. Under the same conditions and if $\mathfrak{p}$ is any maximal r-prime ideal belonging to $\mathfrak{a}$, then $\mathfrak{a} \subset \mathfrak{a}_r(\mathfrak{p})$. Likewise, if $\mathfrak{a}$ has at least two different maximal l-prime ideals belonging to $\mathfrak{a}$, then $\mathfrak{a} \subset \mathfrak{a}_l(\mathfrak{p}')$ for any maximal l-prime ideal $\mathfrak{p}'$ belonging to $\mathfrak{a}$.

Next we assume that $\mathfrak{p}$ is the only maximal r-prime ideal belonging to $\mathfrak{a}$. As we have seen $\mathfrak{a} \subseteq \mathfrak{a}_r(\mathfrak{p})$. But now $\mathfrak{p}$ contains all and only those elements which are r-related to $\mathfrak{a}$. Suppose that c is an arbitrary element of $\mathfrak{a}_r(\mathfrak{p})$. Then there exists an element d of $S_{\mathfrak{p}}$ such that $cxd \equiv 0(\mathfrak{a})$ for all $x \in R$. As $d \not\equiv 0(\mathfrak{p})$, d is r-unrelated to $\mathfrak{a}$. This means $c \equiv 0(\mathfrak{a})$ and $\mathfrak{a}_r(\mathfrak{p}) \subseteq \mathfrak{a}$. It follows that $\mathfrak{a} = \mathfrak{a}_r(\mathfrak{p})$. Analogously, if $\mathfrak{p}'$ is the only maximal l-prime ideal belonging to $\mathfrak{a}$, then $\mathfrak{a} = \mathfrak{a}_l(\mathfrak{p}')$.

If $\mathfrak{p}_1$ is a maximal r-prime ideal belonging to $\mathfrak{a}_r(\mathfrak{p})$, then $\mathfrak{p}_1 \subseteq \mathfrak{p}$. Indeed, if $\mathfrak{p}_1 \not\subseteq \mathfrak{p}$, then $\mathfrak{p}_1$ would contain at least one element of $S_{\mathfrak{p}}$, that is, an element r-unrelated to $\mathfrak{a}_r(\mathfrak{p})$, which is impossible, since $\mathfrak{p}_1$ is r-related to

$\mathfrak{a}_r(\mathfrak{p})$. If $\mathfrak{p}_2$ is a maximal l-prime ideal belonging to $\mathfrak{a}_l(\mathfrak{p}')$, then $\mathfrak{p}_2 \subseteq \mathfrak{p}'$. The proof is similar.

Theorem 2. The principal r-component $\mathfrak{a}_r(\mathfrak{p})$ is contained in every ideal $\mathfrak{d}$, such that $\mathfrak{a} \subseteq \mathfrak{d}$ and all maximal r-prime ideals belonging to $\mathfrak{d}$ are contained in $\mathfrak{p}$. The principal l-component $\mathfrak{a}_l(\mathfrak{p}')$ is contained in every ideal $\mathfrak{d}'$, such that $\mathfrak{a} \subseteq \mathfrak{d}'$ and all maximal l-prime ideals belonging to $\mathfrak{d}'$ are contained in $\mathfrak{p}'$.

Proof. If $\mathfrak{p}=R$ resp. $\mathfrak{p}'=R$, then $\mathfrak{a}_r(\mathfrak{p})=\mathfrak{a}$ resp. $\mathfrak{a}_l(\mathfrak{p}')=\mathfrak{a}$ (definition 7), and all is clear. Now let $\mathfrak{p} \neq R$ and $\mathfrak{d}$ contain $\mathfrak{a}$, such that the maximal r-prime ideals belonging to $\mathfrak{d}$ are contained in $\mathfrak{p}$. If r is an arbitrary element of $\mathfrak{a}_r(\mathfrak{p})$, there exists an element s of $S_\mathfrak{p}$ such that $rxs \equiv 0(\mathfrak{a})$ for all $x \in R$. Then $rxs \equiv 0(\mathfrak{d})$, s being r-unrelated to $\mathfrak{d}$, as all maximal r-prime ideals belonging to $\mathfrak{d}$ are contained in $\mathfrak{p}$. Therefore $r \equiv 0(\mathfrak{d})$, which implies $\mathfrak{a}_r(\mathfrak{p}) \subseteq \mathfrak{d}$. We can also show, that $\mathfrak{a}_l(\mathfrak{p}') \subseteq \mathfrak{d}'$ for every ideal $\mathfrak{d}'$ satisfying the conditions of the theorem.

From the above it follows, that $\mathfrak{a}_r(\mathfrak{p})$ is the "least" ideal containing $\mathfrak{a}$ such that all maximal r-prime ideals belonging to $\mathfrak{a}_r(\mathfrak{p})$ are contained in $\mathfrak{p}$. The ideal $\mathfrak{a}_r(\mathfrak{p}')$ has a similar property.

Theorem 3. $\mathfrak{a}$ is the intersection of all its principal r-components $\mathfrak{a}_r(\mathfrak{p})$.

Proof. If $\mathfrak{p}=R$, then $\mathfrak{a}_r(\mathfrak{p})=\mathfrak{a}$ according to definition 7. Now we suppose $\mathfrak{p} \neq R$. Since $\mathfrak{a}$ is contained in every principal r-component of $\mathfrak{a}$, $\mathfrak{a}$ is also contained in the intersection of these ideals. Conversely, let $a \in \mathfrak{a}_r(\mathfrak{p})$ for every maximal r-prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$. For any ideal $\mathfrak{p}$ we have $axs \equiv 0(\mathfrak{a})$ for all $x \in R$ and some s not in $\mathfrak{p}$.

Let $\mathfrak{d}$ be the ideal consisting of the elements d such that $axd \equiv 0(\mathfrak{a})$ for all $x \in R$. Then, for every maximal r-prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$, $\mathfrak{d}$ contains an element d not in $\mathfrak{p}$, as $a \in \mathfrak{a}_r(\mathfrak{p})$. Thus $\mathfrak{d}$ is not contained in any maximal r-prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$.

According to theorem 1, $\mathfrak{d}$ can not be r-related to $\mathfrak{a}$. Therefore, $\mathfrak{d}$ contains at least one element, say d', r-unrelated to $\mathfrak{a}$. Then $axd' \equiv 0(\mathfrak{a})$ for all $x \in R$ and d' is r-unrelated to $\mathfrak{a}$. It follows that $a \equiv 0(\mathfrak{a})$.

We have proved: $a \in \mathfrak{a}_r(\mathfrak{p})$ for every maximal r-prime ideal belonging to $\mathfrak{a}$ implies $a \in \mathfrak{a}$. This proves the theorem.

Remark. Evidently, we can prove $\mathfrak{a}$ is the intersection of all its principal l-components $\mathfrak{a}_l(\mathfrak{p}')$ in a similar way.

3. *The isolated component ideal of an ideal*

The principal r-components of $\mathfrak{a}$ are important examples of the isolated r-components of $\mathfrak{a}$, defined as follows:

Definition 8. The isolated r-component ideal of $\mathfrak{a}$, generated by the m-system S, consists of all and only those elements i of R, such that there exists an element s of S with $ixs \equiv 0(\mathfrak{a})$ for all $x \in R$.

The isolated l-component ideal of $\mathfrak{a}$, generated by the m-system S', consists of all and only those elements j of R, such that there exists an element s' of S' with $s'yj \equiv 0(\mathfrak{a})$ for all $y \in R$.

The isolated r-component resp. l-component ideal of $\mathfrak{a}$ may be denoted by $(\mathfrak{a}_S)_r$ resp. $(\mathfrak{a}_{S'})_l$ if S resp. S' is the generating m-system. Evidently $(\mathfrak{a}_S)_r = R$ if and only if S contains an element of $\mathfrak{a}$. For if S contains the element a of $\mathfrak{a}$, then $rxa \equiv 0(\mathfrak{a})$ for all $x \in R$ and every element r of R, so $(\mathfrak{a}_S)_r = R$. Conversely, if $(\mathfrak{a}_S)_r = R$, then for an element s_1 of S there exists an element s_2 of S such that $s_1xs_2 \equiv 0(\mathfrak{a})$ for all $x \in R$. As S is an m-system, $s_1, s_2 \in S$ imply the existence of an element x_1 of R with $s_1x_1s_2 \in S$. Since $s_1x_1s_2 \equiv 0(\mathfrak{a})$, we conclude that S contains an element of $\mathfrak{a}$. We have also: $(\mathfrak{a}_{S'})_l = R$ if and only if S' contains an element of $\mathfrak{a}$. — If $\mathfrak{b} = (\mathfrak{a}_S)_r$ of $\mathfrak{a}$, $\mathfrak{c} = (\mathfrak{b}_{S_1})_r$ of $\mathfrak{b}$, and S_2 is the least m-system, containing both S and S_1, then $\mathfrak{c} = (\mathfrak{a}_{S_2})_r$ is also an isolated r-component ideal of $\mathfrak{a}$. If b is an arbitrary element of $\mathfrak{b}$, then $bxs \equiv 0(\mathfrak{a})$ for all $x \in R$ and some element s of S. If c is an arbitrary element of $\mathfrak{c}$, then $cys_1 \equiv 0(\mathfrak{b})$ for all $y \in R$ and some element s_1 of S_1. From $cys_1 \in \mathfrak{b}$ it follows that $cys_1xs' \equiv 0(\mathfrak{a})$ for all $x, y \in R$ and some $s' \in S$. But $s_1 \in S_1 \subseteq S_2$ and $s' \in S \subseteq S_2$, so there exists an element t of R with $s_1ts' \in S_2$, S_2 being an m-system. Now $cys_1ts' \equiv 0(\mathfrak{a})$ for all $y \in R$ implies $c \in (\mathfrak{a}_{S_2})_r$, which completes the proof. In a similar way we can show that the isolated l-component ideal of an isolated l-component ideal is an isolated l-component ideal of the original ideal. In the special case that S_1 is an m-system containing S we have $S_2 = S_1$ and $(\{(\mathfrak{a}_S)_r\}_{S_1})_r = (\mathfrak{a}_{S_1})_r$ and likewise for the isolated l-component ideal.

An isolated r-component ideal $(\mathfrak{a}_S)_r$ can be generated by different m-systems, generally speaking. Now every element of S is r-unrelated to $(\mathfrak{a}_S)_r$, which can be proved in the general case in the same way as in the special case of the principal r-components of $\mathfrak{a}$. Therefore the m-system S^*, consisting of the elements which are r-unrelated to $(\mathfrak{a}_S)_r$, contains the m-system S, and we may write $(\{(\mathfrak{a}_S)_r\}_{S^*})_r = (\mathfrak{a}_{S^*})_r$.

Now we can prove: the m-system S^* of all elements r-unrelated to $(\mathfrak{a}_S)_r$ is the greatest m-system generating the isolated r-component ideal $(\mathfrak{a}_S)_r$ of $\mathfrak{a}$. If i is an arbitrary element of $(\mathfrak{a}_S)_r$, then $ixs \equiv 0(\mathfrak{a})$ for all $x \in R$ and an element s of S; but $S \subseteq S^*$, and $s \in S^*$, therefore $i \in (\mathfrak{a}_{S^*})_r$ and we see that $(\mathfrak{a}_S)_r \subseteq (\mathfrak{a}_{S^*})_r$. Conversely, if j is an arbitrary element of $(\mathfrak{a}_{S^*})_r$, then $j \in (\{(\mathfrak{a}_S)_r\}_{S^*})_r$, which implies $jys^* \equiv 0(\mathfrak{a}_S)_r$ for all $y \in R$ and a suitable element s^* of S^*. Now s^* is r-unrelated to $(\mathfrak{a}_S)_r$, so $j \equiv 0(\mathfrak{a}_S)_r$. Then $(\mathfrak{a}_{S^*})_r \subseteq (\mathfrak{a}_S)_r$, consequently $(\mathfrak{a}_S)_r = (\mathfrak{a}_{S^*})_r$. Evidently S^* is the greatest m-system with this property. As to the isolated l-component ideal $(\mathfrak{a}_{S'})_l$ we obtain: the m-system S'^* of all elements l-unrelated to $(\mathfrak{a}_{S'})_l$ is the greatest m-system generating $(\mathfrak{a}_S')_l$ of $\mathfrak{a}$.

As we have seen, the elements r-related to $(\mathfrak{a}_S)_r$ are spread over the maximal r-prime ideals belonging to $(\mathfrak{a}_S)_r$ (theorem 1). It follows, that the m-system S^* consisting of the elements r-unrelated to $(\mathfrak{a}_S)_r$ is determined uniquely by the maximal r-prime ideals belonging to $(\mathfrak{a}_S)_r$. Now if $(\mathfrak{a}_S)_r$ and $(\mathfrak{a}_{S_1})_r$ are two isolated r-component ideals of $\mathfrak{a}$, such that the maximal r-prime ideals belonging to $(\mathfrak{a}_S)_r$ are the same as those ones belonging to $(\mathfrak{a}_{S_1})_r$, the m-systems S^* resp. $S_1{}^*$ of elements r-unrelated to $(\mathfrak{a}_S)_r$ resp. $(\mathfrak{a}_{S_1})_r$ are identical, consequently $(\mathfrak{a}_S)_r=(\mathfrak{a}_{S_1})_r$, (definition 8). Two isolated r-component (l-component) ideals with the same maximal r-prime (l-prime) ideals are identical.

Theorem 4. Every maximal l-prime ideal belonging to $(\mathfrak{a}_S)_r$ is contained in a maximal l-prime ideal belonging to $\mathfrak{a}$.

Proof. In order to prove this theorem, we only need to show that if s is an element l-unrelated to $\mathfrak{a}$, then s is l-unrelated to $(\mathfrak{a}_S)_r$. Now suppose that $sxc \equiv 0(\mathfrak{a}_S)_r$ for all $x \in R$, then there exists an element $t \in S$ such that $sxcyt \equiv 0(\mathfrak{a})$ for all $x, y \in R$. As s is l-unrelated to $\mathfrak{a}$, we have $cyt \equiv 0(\mathfrak{a})$, which implies $c \equiv 0(\mathfrak{a}_S)_r$. Therefore s is l-unrelated to $(\mathfrak{a}_S)_r$.

It may be remarked that every maximal r-prime ideal belonging to $(\mathfrak{a}_{S'})_l$ is contained in a maximal r-prime ideal belonging to $\mathfrak{a}$.

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